Asymptotische Statistik
Winter term 2018/19
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Exercise sheet 2

Exercise 1

Show Lemma 2.0.2 from the lecture:
Assume that

(i) $\Theta \subset \mathbb{R}^d$ is compact,

(ii) $Q : \Theta \to \mathbb{R}, \theta \mapsto Q(\theta)$ is a continuous map,

(iii) $Q(\theta) > Q(\theta_0)$ for all $\theta \in \Theta$ with $\theta \neq \theta_0$.

Then:

$$\inf_{\|\theta-\theta_0\|>\varepsilon} Q(\theta) > Q(\theta_0)$$

for all $\varepsilon > 0$. **(2 points)**

Exercise 2

Let $\Theta = [a, b] \subset \mathbb{R}$ and consider the deterministic function $M : \Theta \to \mathbb{R}$ as well as the random functions $M_n : \Theta \to \mathbb{R}$ for $n \in \mathbb{N}$. Assume that

(i) M is continuous,

(ii) M_n is monotonically increasing for all $n \in \mathbb{N}$,

(iii) $M_n(\theta) = M(\theta) + o_{\mathbb{P}}(1)$ for all $\theta \in \Theta$.

Show that

$$\sup_{\theta\in\Theta} |M_n(\theta) - M(\theta)| = o_{\mathbb{P}}(1),$$

i.e. uniform convergence in the sense of Theorem 2.0.1. **(4 points)**

Exercise 3

Let $Z_1, \dots, Z_n$ be $\mathbb{R}$-valued random variables with distribution function F. Denote by $\theta_0 \in \Theta = \mathbb{R}$ a median of F, i.e. $F(\theta_0) = \frac{1}{2}$. Define

$$Q : \Theta \to \mathbb{R}, \theta \mapsto Q(\theta) := \mathbb{E}[|Z_1 - \theta| - |Z_1|],$$

$$Q_n : \Theta \to \mathbb{R}, \theta \mapsto Q_n(\theta) := \frac{1}{n}\sum_{i=1}^{n} |Z_i - \theta| - |Z_i| \quad \text{for } n \in \mathbb{N}.$$

Assume that for all $\theta_1, \theta_2 \in \Theta$ with $\theta_1 < \theta_0 < \theta_2$ it holds that $F(\theta_1) < \frac{1}{2} < F(\theta_2)$.
In the lecture it was shown that Q fulfills the identifiability condition. To show that $\hat{\theta}_n := \arg\min_{\theta\in\Theta} Q_n(\theta)$ is a consistent estimator of the mean, prove that Q_n and Q fulfill the uniform convergence condition in the sense of Theorem 2.0.1. **(4 points)**

Exercise 4

Consider the following regression model.

$$Y_i = \theta_{0,1} + \theta_{0,2}X_i + \varepsilon_i, \quad i \in \mathbb{N}$$

with $X_i \overset{iid}{\sim} U[-1, 1], \varepsilon_i \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$ for $i \in \mathbb{N}$ while the X_i and ε_i are independent and $\sigma > 0$ is fixed. Let $\theta_0 = (\theta_{0,1}, \theta_{0,2}) \in \Theta$ where $\Theta \subseteq \mathbb{R}^2$ is compact.
Show that

$$\hat{\theta}_n := \underset{(\theta_1,\theta_2)\in\Theta}{\arg\min} \frac{1}{n}\sum_{i=1}^{n}(Y_i - \theta_1 - \theta_2 X_i)^2$$

is a consistent M-estimator for θ_0. **(4 points)**

The solutions are to be handed in at the designated **box number 03** on the first floor of INF 205 (Mathematikon) in front of the Dekanat by **Tuesday, October 30th 2018** before 9 c.t..