Asymptotische Statistik
Winter term 2018/19
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Exercise sheet 3

Exercise 1

We continue Exercise 4 from Exercise sheet 2 and therewith all notation and assumtions made. Additionally assume that $\theta_0 \in \text{int}\Theta$.
Show asymptotic normality of the given estimator $\hat{\theta}_n$. **(2 points)**

Exercise 2

For $i \in \mathbb{N}$ let X_i be *iid* exponentially distributed with parameter $\lambda_0 \in [0, L]$ for an $L > 0$. Show that the ML-estimator given by

$$\hat{\lambda}_n := \underset{\lambda \in [0,L]}{\arg\max} \prod_{i=1}^{n} f_\lambda(X_i)$$

is a consistent estimator for λ_0, where $f_\lambda : x \mapsto \lambda e^{-\lambda x}$ denotes the density of an expontentially distributed random variable with parameter λ.
Additionally prove that $\hat{\lambda}_n$ is asymptotically normal. Define an estimator for the variance and show its consistency. **(3 points)**

Exercise 3

Consider the setup and all assumptions including their notation of Theorem 3.0.1 and another estimator $\check{\theta}_n$ of θ_0 with $\|\check{\theta}_n - \theta_0\| = O_{\mathbb{P}}(n^{-\frac{1}{2}})$. Define

$$\tilde{\theta}_n := \check{\theta}_n - H_n^{-1}(\check{\theta}_n) S_n(\check{\theta}_n)$$

for $n \in \mathbb{N}$. Show that

$$\tilde{\theta}_n = \hat{\theta}_n + o_{\mathbb{P}}(n^{-\frac{1}{2}})$$

and conclude that $\sqrt{n}(\tilde{\theta}_n - \theta_0)$ has the same asymptotically normal limit as $\sqrt{n}(\hat{\theta}_n - \theta_0)$. **(4 points)**

Exercise 4

Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, 1)$ for $\mu \geq 0$. Calculate the ML-esimator $\hat{\mu}_n$ of μ and show that $\sqrt{n}(\hat{\mu}_n - \mu)$ is not for all μ asymptotically normal. Comment on why this holds. **(2 points)**

Exercise 5

Show the following modification of the delta method (Lemma 0.8):
Let $\phi : U \to \mathbb{R}^q$ be a continuously differentiable function that is defined on a neighbourhood $U \subset \mathbb{R}^p$ of a point $x \in \mathbb{R}^p$. For a sequence of $\mathbb{R}^p$-valued random variables X_n and a sequence x_n in $\mathbb{R}^p$ with $x_n \to x$ assume that

$$\sqrt{n}\left(X_n - x_n\right) \quad \overset{d}{\to} \quad N\left(0, \Sigma\right).$$

Show that it holds that

$$\sqrt{n}\left(\phi\left(X_n\right) - \phi\left(x_n\right)\right) \quad \overset{d}{\to} \quad N\left(0, \phi'\left(x\right)\Sigma\phi'\left(x\right)^T\right).$$

(4 points)

The solutions are to be handed in at the designated **box number 03** on the first floor of INF 205 (Mathematikon) in front of the Dekanat by **Tuesday, November 6th 2018** before 9 c.t..