Asymptotische Statistik
Winter term 2018/19
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Exercise sheet 8

Exercise 1

Do Exercise 11 from the lecture:
Show that

$$H(\varepsilon, \mathcal{F}, \|\cdot\|_\infty) \leq C\varepsilon^{-1}$$

for $\mathcal{F} = \{f : [a, b] \to [c, d] \text{ with } |f'| \leq \rho\}$ for some constant C depending only on a, b, c, d and ρ.

(4 points)

Exercise 2

We want to prove Theorem 8.1.1:
We consider the following model.

$$Y_i = m_0(X_i) + \theta_0^T Z_i + \epsilon_i, \quad i = 1, \dots, n$$

with $\mathbb{E}[\varepsilon_i|X_i, Z_i] = 0$ and (X_i, Z_i, ϵ_i) iid. Assume that $X_i \in [0, 1]$ and define the following estimator.

$$(\hat{m}_n, \hat{\theta}_n) = \underset{(m,\theta)}{\arg\min} \frac{1}{n}\sum_{i=1}^n (Y_i - m(X_i) - \theta Z_i)^2 + \lambda_n^2 I^2(m)$$

with $I^2(m) = \int_0^1 m^{(k)}(x)^2 \mathrm{d}x$. Make the assumptions (A1)-(A6) from chapter 8. Then it holds that

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = \frac{1}{\sqrt{n}} \frac{1}{\mathbb{E}[R^2]} \sum_{i=1}^n \epsilon_i R_i + o_{\mathbb{P}}(1)$$

with $R_i = Z_i - h(X_i)$.

(i) Show that $T_1 + T_2 + T_3 - T_4 = 0$ with

$$T_1 = -\frac{1}{n}\sum_{i=1}^n \epsilon_i R_i,$$

$$T_2 = (\hat{\theta}_n - \theta_0)\frac{1}{n}\sum_{i=1}^n Z_i(Z_i - h(X_i)),$$

$$T_3 = \frac{1}{n}\sum_{i=1}^n (\hat{m}_n(X_i) - m_0(X_i))(Z_i - h(X_i)),$$

$$T_4 = \lambda_n^2 \int_0^1 \hat{m}^{(k)}(x) h^{(k)}(x) \, \mathrm{d}x$$

where $h(x) = \mathbb{E}[Z|X_x]$ and show that

$$\sqrt{n}(\hat{\theta}_n - \theta_0)\frac{1}{n}\sum_{i=1}^n Z_i(Z_i - h(X_i)) = \frac{1}{\sqrt{n}}\sum_{i=1}^n \epsilon_i R_i - \sqrt{n}(T_3 + T_4).$$

(ii) Prove that

$$\frac{1}{n}\sum_{i=1}^n Z_i(Z_i - h(X_i)) \xrightarrow{\mathbb{P}} \mathbb{E}[R^2]$$

for $n \to \infty$.

(iii) Assume that Lemma 6.3.2 also holds for $\varepsilon_i = R_i$ and show that

$$T_3 = o_{\mathbb{P}}(n^{-\frac{1}{2}}),$$
$$T_4 = o_{\mathbb{P}}(n^{-\frac{1}{2}}).$$

(3+1+2 points)

Exercise 3

Do Exercise 13 from the lecture.
Let ψ as in Lemma 9.2.3 and define $\phi(x) := \sigma\psi(\tau x)$ for $\sigma \in (0,1)$, $\tau > 0$.

(i) Show that

$$\|X\|_\psi \leq \frac{\|X\|_\phi}{\sigma\tau} \leq \frac{\|X\|_\psi}{\sigma}$$

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(ii) Show that for any ψ as above there exist a $\sigma \in (0,1)$ and a $\tau > 0$, such that there is a $\tilde{c} > 0$ with

$$\phi(x)\phi(y) \leq \phi(\tilde{c}xy)$$

for all $x, y \geq 1$ and $\phi(1) \leq \frac{1}{2}$.

(iii) Finish the proof of Lemma 9.2.3. **(1+1+2 points)**

The solutions are to be handed in at the designated **box number 03** on the first floor of INF 205 (Mathematikon) in front of the Dekanat by **Tuesday, December 11th 2018** before 9 c.t..