Probability Theory II
Summer term 2018
Lecturer: Prof. Dr. Enno Mammen
Assistant: Marilena Müller

UNIVERSITÄT HEIDELBERG
ZUKUNFT SEIT 1386

Exercise sheet 1

Exercise 1. Let $\mathcal{F}$ be a space of functions $D \to \mathbb{R}$. Recall the supremum norm given by $\|f\|_\infty := \sup_{x\in D} |f(x)|$ for $f \in \mathcal{F}$.

(i) Show that the set $\mathcal{C}[0,1]$ of all continuous functions $[0,1] \to \mathbb{R}$ endowed with the supremum norm $\|\cdot\|_\infty$ is a separable space.

(ii) Consider the space of all right-continuous functions $[0,1] \to \mathbb{R}$ with existing left side limits, i.e.

$$\mathcal{D}[0,1] = \Big\{ f : [0,1] \to \mathbb{R} : \forall x_0 \in [0,1] \lim_{x \searrow x_0} f(x) = f(x_0), \lim_{x \nearrow x_0} f(x) \text{ exists} \Big\}.$$

Show that $(\mathcal{D}[0,1], \|\cdot\|_\infty)$ is not separable. **(2+2 points)**

Exercise 2. Let $\{(S_i, d_i)\}_{i\in\mathbb{N}}$ be a family of Polish spaces. Define $S := \prod_{i\in\mathbb{N}} S_i$ and $d : S \times S \to \mathbb{R}$ with

$$d(x,y) := \sum_{i\in\mathbb{N}} 2^{-i} \wedge d_i(x_i, y_i)$$

where $x = (x_i)_{i\in\mathbb{N}}, y = (y_i)_{i\in\mathbb{N}} \in S$. Show that (S,d) is a Polish space and that d generates the same topology on S as the topological basis

$$\Big\{ \bigcap_{j\in J} \pi_j^{-1}(O_j) : O_j \subseteq S_j \text{ open}, J \subset \mathbb{N} \text{ finite} \Big\}$$

where $\pi_k : S \to S_k, (x_i)_{i\in\mathbb{N}} \mapsto x_k$ denote the projection maps. **(2+3 points)**

Exercise 3. Let (S,d) be a complete metric space and $A \subset S$ a subset. Show that A is totally bounded if and only if A is relatively compact. Recall that

- A subset D of a metric space is called *totally bounded* if for all $\varepsilon > 0$ there exists a finite set of elements $\{x_1, \ldots, x_k\} \subset D$ such that

$$D \subset \bigcup_{j=1}^{k} B_\varepsilon(x_j)$$

  where $B_\delta(y)$ denotes the open δ-ball around an element y of the underlying metric space.

- A subset D of a metric space is called *relatively compact* if its closure is compact in the underlying metric space. **(4 points)**

Exercise 4. For a random variable Y denote by $\mathbb{P}^Y$ the induced probability measure. We call a family $\mathcal{P}$ of probabiliy measures on a topological space *(uniformly) tight* if for all $\varepsilon > 0$ there exists a compact set K_ε such that $P(K_\varepsilon) \geq 1 - \varepsilon$ for all $P \in \mathcal{P}$.

(i) Let $(X_i)_{i\in I}$ be a family of $\mathbb{R}$-valued random variables with $\sup_{i\in I} \mathbb{E}[|X_i|] < \infty$. Show that $\{\mathbb{P}^{X_i}\}_{i\in I}$ is (uniformly) tight.

(ii) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}$-valued random variables with $X_n \xrightarrow{d} X$ for a random variable X. Show that $\{\mathbb{P}^{X_n}\}_{n\in\mathbb{N}}$ is (uniformly) tight. **(1+2 points)**

The solutions are to be handed in at the designated **box number 14** on the first floor of INF 205 (Mathematikon) in front of the Dekanat by **Thursday, 26th April 2018** before 9 c.t..